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A Characterization of Weighted Peetre K-Functionals

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The goal of this paper is to prove the equivalence of properly defined moduli of functions and the Peetre K-functional $K(t^k, f) = \inf\{\|f - g\|_p + t^k \|W^k g^{(k)}\|_p : g\}$ for a wide class of weights w. The paper continues the investigations of Ditzian [3] and Totik [12] and shows that the moduli used by both authors are in many cases equivalent to the moduli introduced in [7]. Proximate inequalities between the K-functionals for different weights are derived.

1. INTRODUCTION

We deal with functions defined on the (finite or infinite) interval $[a, b]$. Let $L_p[a, b]$ $(1 \leqslant p \leqslant \infty)$ be the set of all classes of measurable functions f for which

$$\|f\|_p = \|f\|_{p[a,b]} = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p} < \infty,$$

let $C[a, b]$ be the set of all continuous functions in $[a, b]$ with a norm

$$\|f\|_\infty = \|f\|_{\infty[a,b]} = \sup\{|f(x)| : x \in [a, b]\},$$

and let $W_p^k(w)$ $(1 \leqslant p \leqslant \infty$, k natural) be the set of all functions which are locally absolutely continuous together with $g', ..., g^{(k-1)}$ and $\|W^k g^{(k)}\|_p < \infty$, where the weight w is continuous and locally positive in $[a, b]$. Here and throughout "locally" means that the property is fulfilled in

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every subinterval $[a', b']$ $(a<a'<b'<b)$ of the interval $[a, b]$. The weighted Peetre K-functional for the function f is given by

$$K(t^k, f) = K(t^k, f; L_p, W_p^k(w)) \\ = \inf\{\|f-g\|_p + t^k \|w^k g^{(k)}\|_p : g \in W_p^k(w)\}. \tag{1.1}$$

Let us underline that we require $f \in L_p + W_p^k(w)$, i.e., $f = f_0 + f_1$ for some $f_0 \in L_p$, $f_1 \in W_p^k(w)$, and therefore $f \in L_{p,\mathrm{loc}}[a, b]$; but in general f will not belong to $L_p[a, b]$.

The weighted K-functional has proved useful in the characterization of many approximating processes. More precisely, the equivalence

$$\|f - M_n f\|_{p[a,b]} = O(n^{-\beta}) \Leftrightarrow K(t^k, f; L_p[a, b], W_p^k(w)) = O(t^\alpha), \tag{1.2}$$

$0 < \alpha < k$, holds true when:

(a) M_n is the operator of best approximation in $L_p[a, b]$ by algebraic polynomials of degree n, $w(x) = \sqrt{(b-x)(x-a)}$, $1 \leqslant p \leqslant \infty$, $\beta = \alpha$, natural k;

(b) $M_n f$ are Bernstein polynomials, $[a, b] = [0, 1]$, $w(x) = \sqrt{(x - x^2)}$, $p = \infty$, $\beta = \alpha/2$, $k = 2$ (Berens and Lorentz [1], Ditzian [4]);

(c) $M_n f$ are Kantorovich polynomials, $[a, b] = [0, 1]$, $w(x) = \sqrt{(x - x^2)}$, $1 \leqslant p \leqslant \infty$, $\beta = \alpha/2$, $k = 2$ (Grundmann [6], Muller [10]);

(d) $M_n f$ are Szasz–Mirakjan ($w(x) = \sqrt{x}$) or Baskakov ($w(x) = \sqrt{(x + x^2)}$) operators, $[a, b] = [0, \infty)$, $p = \infty$, $\beta = \alpha/2$, $k = 2$ (Totik [13]).

Many other examples for the validity of (1.2) with different M_n can be given.

But, when we want to calculate the degree of approximation of a given function f, the equivalence (1.2) is not very useful—the class of functions for which one can evaluate directly the infimum in (1.1) is rather narrow. Fortunately, the K-functionals are equivalent to moduli of smoothness which are easier to compute. In the case $w(x) \equiv 1$ the equivalence

$$c(k)\, \omega_k(f; t)_p \leqslant K(t^k, f; L_p, W_p^k(1)) \leqslant c(k)\, \omega_k(f; t)_p \tag{1.3}$$

is well known, where the moduli of smoothness are given by

$$\omega_k(f; t)_p = \sup\{\|\Delta_h^k f\|_p : 0 < h \leqslant 1\}.$$

Equivalence (1.3) was extended with suitably defined moduli for different types of weights by Ditzian [3] ($w(x) = x^\alpha$, $x \in [0, 1]$, natural k) and Totik [12] ($k = 2$ and w twice locally differentiable). In this paper we use other kinds of moduli to establish an analog of (1.3) for the K-functionals

(1.1). These moduli were introduced by the author [7] to characterize the best algebraic approximations and the approximations by Bernstein polynomials.

During the preparation of the manuscript we learned that Ditzian and Totik also generalized in [5] the results from [3] and [12]. Many applications of the weighted Peetre K-functionals in approximation theory are also given in [5].

The moduli we shall use (see [7, 8, 9]) are given by

$$\tau_k(f; \psi(t))_{q,p} = \|\omega_k(f, \cdot; \psi(t, \cdot))_q\|_p,$$

where

$$\omega_k(f, x; \psi(t, x))_q = \left[(2\psi(t, x))^{-1} \int_{-\psi(t,x)}^{\psi(t,x)} |\Delta_h^k f(x)|^q \, dh \right]^{1/q} \qquad (1 \leqslant q < \infty),$$

$$\omega_k(f, x; \psi(t, x))_\infty = \sup\{|\Delta_h^k f(x)| : |h| \leqslant \psi(t, x)\},$$

and $\Delta_h^k f(x) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x+ih)$ if $x, x+kh \in [a, b]$ and $\Delta_h^k f(x) = 0$ otherwise. Here ψ is a continuous positive function of x in $[a, b]$ for any $t \in (0, t_0]$.

The main result of the paper is

$$\begin{aligned} c(k, w)\, \tau_k(f; \psi(t))_{p,p} &\leqslant K(t^k, f; L_p, W_p^k(w)) \\ &\leqslant c(k, w)\, \tau_k(f; \psi(t))_{p,p}, \end{aligned} \tag{1.4}$$

where the connection between w and ψ is given by (3.11).

The paper is organized as follows. In Section 3 we describe different types of weights and give some of their properties. Inequalities for moduli of differentiable functions and the proof of the first inequality in (1.4) are given in Section 4. Following the ideas in [9] we construct appropriate intermediate functions and complete the proof of (1.4) in Section 5. Various kinds of properties of the moduli are derived in Section 6 as a consequence of the previous results. Proximate inequalities between the K-functionals corresponding to different weights are obtained in Section 7.

2. Priliminaries

In the paper $1 \leqslant p, q \leqslant \infty$, $1/p + 1/p' = 1$; $\lambda, t, \eta = \text{const} > 0$; k is natural; μ is a fixed $C^\infty(\mathbb{R})$ function such that $\mu(x) = 0$ for $x \leqslant 0$, $\mu(x) = 1$ for $x \geqslant 1$, and $0 < \mu(x) < 1$ for $0 < x < 1$; c denotes a positive number which may be different at each occurence. The exact dependence of c on the other parameters is explicitly given. With A, A', $A_1, \ldots$ we denote constants preserving their values throughout the paper.

Two functions, v and u, are associated with the weight w in neighbourhoods of the end-points a and b. Let a and b be finite. Consider a neighbourhood $[a, d]$ of a or $[d, b]$ of b; we denote $v(x) = x/w(a+x)$ for $x \in (0, d-a]$ or $v(x) = x/w(b-x)$ for $x \in (0, b-d]$, respectively. u is the inverse function to v, i.e., $u(v(x)) = v(u(x)) = x$. In the case u will be used a and b will be finite, v will be continuous, strictly monotone, and $v(0) = 0$. For $a = 0$ the functions u and w are connected by

$$u(x) = v(u(x))\, w(u(x)) = xw(u(x)).$$

For infinite end points we set $v(x) = x/w(x)$ $(b = \infty)$ or $v(x) = x/w(-x)$ $(a = -\infty)$ for $x \in [d, \infty)$, $d > 0$.

Different forms of Minkowski's and Holder's inequalities will often be used without explicit mention.

Let H_n be the set of all algebraic polynomials of degree not greater than n and let

$$E_n(f)_{p[a,b]} = \inf\{\|f - Q\|_{p[a,b]} : Q \in H_n\}$$

denote the best algebraic approximation of f in $L_p[a, b]$.

The inequality

$$E_{k-1}(f)_{p[a,b]} \leqslant c(k)\, \omega_k(f; (b-a)/k)_{p[a,b]}, \tag{2.1}$$

known as Whitney's theorem, was proved by H. Whitney [14] for $p = \infty$ and $f \in C[a, b]$ and was extended by Y. A. Brudnii [2] for $1 \leqslant p < \infty$ and $f \in L_p[a, b]$ ($[a, b]$ finite).

We assume the properties of $\omega_k(f; t)_p$ are known.

LEMMA 2.1.

$$\tau_k(f+g; \psi(t))_{q,p} \leqslant \tau_k(f; \psi(t))_{q,p} + \tau_k(g; \psi(t))_{q,p}. \tag{2.2}$$

$$\tau_k(\alpha f; \psi(t))_{q,p} = |\alpha|\, \tau_k(f; \psi(t))_{q,p} \qquad (\text{real } \alpha). \tag{2.3}$$

$$\tau_k(f; \psi(t))_{q,p} \leqslant \tau_k(f; \psi(t))_{r,p} \qquad \text{if} \quad 1 \leqslant q \leqslant r \leqslant \infty. \tag{2.4}$$

This lemma follows directly from the definition of τ moduli.

LEMMA 2.2. *If we assume that $f(z) = 0$ in (2.6) or $f^{(k)}(z) = 0$ in (2.7) when z does not belong to $[a, b]$, then*

$$\omega_k(f, x; h_1)_q \leqslant A'\omega_k(f, x; h_2)_q \qquad \textit{if} \quad h_1 \leqslant h_2 \leqslant A'h_1, \tag{2.5}$$

$$\omega_k(f, x; h)_q \leqslant |f(x)| + c(k) \left\{(2kh)^{-1} \int_{-kh}^{kh} |f(x+y)|^q \, dy\right\}^{1/q} \text{ a.e.}, \tag{2.6}$$

$$\omega_k(f, x; h)_q \leqslant c(k)\, h^k \left\{(2kh)^{-1} \int_{-kh}^{kh} |f^{(k)}(x+y)|^q \, dy\right\}^{1/q}$$

$$\textit{if } f^{(k)} \in L_q[x-kh, x+kh]. \tag{2.7}$$

Proof. We get (2.5) and (2.6) from the definition. To prove (2.7) we proceed as follows,

$$\begin{aligned} |\Delta_z^k f(x)| &\leqslant \int_0^z \cdots \int_0^z |f^{(k)}(x+y_1+\cdots+y_k)| \, dy_1 \cdots dy_k \\ &\leqslant c(k)\, z^{k-1} \int_0^{kz} |f^{(k)}(x+y)| \, dy \\ &\leqslant c(k)\, z^{k-1/q} \left[\int_0^{kz} |f^{(k)}(x+y)|^q \, dy\right]^{1/q} \end{aligned}$$

and

$$\begin{aligned} \omega_k(f, x; h)_q^q &= (2h)^{-1} \int_0^h (|\Delta_z^k f(x)|^q + |\Delta_{-z}^k f(x)|^q) \, dz \\ &\leqslant c(k)^q (2h)^{-1} \int_0^h z^{kq-1} \int_0^{kz} (|f^{(k)}(x+y)|^q \\ &\qquad + |f^{(k)}(x-y)|^q) \, dy \, dz \\ &\leqslant c(k)^q (2h)^{-1} \int_0^{kh} (|f^{(k)}(x+y)|^q \\ &\qquad + |f^{(k)}(x-y)|^q) \int_{y/k}^h z^{kq-1} \, dz \, dy \\ &\leqslant c(k)^q h^{kq} (2kh)^{-1} \int_{-kh}^{kh} |f^{(k)}(x+y)|^q \, dy. \quad \blacksquare \end{aligned}$$

Let us denote by $N_k(x)$ the normalized B-spline of degree $k-1$ with nodes $0, 1, ..., k$ (see [11, pp. 134–137]). Then $N_k \in C^{k-2}(\mathbb{R})$, $N_k(x) = 0$ for $x \leqslant 0$ or $x \geqslant k$, $\int_{-\infty}^{\infty} N_k(x)\, dx = 1$, and

$$0 < N_k(x) \leqslant \min\{x^{k-1}; (k-x)^{k-1}\}/(k-1)! \qquad \text{for } x \in (0, k). \tag{2.8}$$

The connection between B-splines and finite differences is given by

$$\Delta_h^k f(x) = h^{k-1} \int_0^{kh} N_k(y/h)\, f^{(k)}(x+y)\, dy. \tag{2.9}$$

The following lemma and (2.4) show that moduli $\tau_k(f; \psi(t))_{p,p}$ can be considered as a generalization of the moduli of smoothness $\omega_k(f; t)_p$.

LEMMA 2.3. *Let $f \in L_p[a, b]$, $\psi(t, x) = t$ for every $x \in [a, b]$, and $0 < t \leqslant (b-a)/(2k)$ if $[a, b]$ is a finite interval. Then*

$$\tau_k(f; t)_{p,p} \leqslant \omega_k(f; t)_p \leqslant c(k)\, \tau_k(f; t)_{1,p}.$$

For a finite interval this is Theorem 3.1 in [8]. The proof for an infinite interval is similar but simpler.

The following embedding lemma will be extensively used (see Lemma 2.1 in [3] or Lemma 2.2 in [9]).

LEMMA 2.4. *Let $[a, b]$ be finite and $g \in W_p^k(1)$. Then for each $j = 0, 1, ..., k$ we have*

$$(b-a)^j \| g^{(j)} \|_{p[a,b]} \leqslant c(k)[\| g \|_{p[a,b]} + (b-a)^k \| g^{(k)} \|_{p[a,b]}].$$

3. BEHAVIOUR OF THE WEIGHT NEAR THE END-POINTS

The weight w is assumed to be continuous and locally positive. Therefore w is bounded from zero and infinity in every closed subinterval of the interior of $[a, b]$. But w may tend to 0 or ∞ at the end-points of the domain. In this section we describe different types of behaviour allowed to the weight. For defining these types we shall work with the neighbourhood $[0, d]$ $(0 < d < \infty)$ of the point 0 and the neighbourhood $[d, \infty)$ $(d < \infty)$ of the point ∞ as representatives of the cases of finite and infinite end-points, respectively. The results for the other end-points can be derived mutatis mutandis.

In the neighbourhood $[0, d]$ of the end-point 0 w will satisfy one of the following three types of conditions.

Type 1. w is non-decreasing, v is strictly increasing in $[0, d]$, and $v(0) = \lim_{x \to 0} v(x) = 0$.

The weights $w(x) = x^\alpha |\log x|^\beta$ ($\alpha = 0$ or $0 < \alpha < 1$ and $\beta \in \mathbb{R}$ or $\alpha = 1$ and $\beta > 0$) are of this type. In this case u is increasing, $u(0) = 0$, and $0 \leqslant w(0) < \infty$.

For $0 < t \leqslant v(d)$ we set

$$\psi(t, x) = tw(x + u(t)) \tag{3.1}$$

assuming that $w(x) = w(d)$ for $x > d$ if w is not defined in $[d, 2d]$.

PROPERTY 1.1. *For $\lambda > 1$ we have $w(\lambda x) < \lambda w(x)$, $v(\lambda x) \leqslant \lambda v(x)$, and $u(\lambda x) \geqslant \lambda u(x)$.*

Proof. $\lambda w(x) = \lambda x / v(\lambda x) > \lambda x / v(\lambda x) = w(\lambda x)$. The same for v. We set $y = u(x)$, $x = v(y)$. Then $v(u(\lambda x)) = \lambda x = \lambda v(y) \geqslant v(\lambda y) = v(\lambda u(x))$ and hence $u(\lambda x) \geqslant \lambda u(x)$. ∎

PROPERTY 1.2. *Let $\lambda \leqslant \frac{1}{2}$ and $|x - y| \leqslant \lambda \psi(t, x)$. Then:*

(a) *$\psi(t, x) \leqslant 2\psi(t, y)$ and $\psi(t, y) \leqslant 1.5 \cdot \psi(t, x)$;*

(b) *$y > x/4$ for $x \geqslant 2u(t)$.*

Proof. For $z \geqslant u(t)$ we have $z = v(z)\, w(z) \geqslant v(u(t))\, w(z) = tw(z)$. Therefore $\psi(t, x) = tw(x + u(t)) \leqslant x + u(t)$ and $|x - y| \leqslant (x + u(t))/2$. Using Property 1.1 we get

$$\psi(t, y) = tw(y + u(t)) \leqslant tw(1.5 \cdot (x + u(t))) < 1.5 \cdot \psi(t, x)$$

and

$$\psi(t, x) = tw(x + u(t)) \leqslant tw(2(y + u(t))) < 2\psi(t, y).$$

If $x > 2u(t)$ then $y \geqslant x - (x + u(t))/2 \geqslant x/4$. ∎

Sometimes we shall require w to satisfy the additional conditions

$$\int_0^x v^k(y)\, y^{-1}\, dy \leqslant A_1 v^k(x) \qquad \text{for every} \quad x \in (0, d], \tag{3.2}$$

or

$$u(\lambda x) \leqslant c(\lambda)\, u(x) \qquad \text{for any} \quad x > 0,\ \lambda \geqslant 1,\ \lambda x \leqslant d. \tag{3.3}$$

One can show that conditions (3.2) and (3.3) are equivalent; that is, w satisfies (3.2) iff it satisfies (3.3), but we shall not make use of this.

PROPERTY 1.3. *Let $\lambda > 1$ and let w satisfy* (3.3). *Then*

$$\lambda \psi(t, x) \leqslant \psi(\lambda t, x) \leqslant c(\lambda)\, \psi(t, x)$$

and

$$c(\lambda)\, \psi(t, x) \leqslant \psi(t/\lambda, x) \leqslant \psi(t, x)/\lambda.$$

Proof. We have $u(t) \leqslant u(\lambda t)$, $w(x+u(t)) \leqslant w(x+u(\lambda t))$, and hence $\lambda\psi(t,x) \leqslant \psi(\lambda t, x)$. From (3.3) and Property 1.1 we get $\psi(\lambda t, x) = \lambda t w(x+u(\lambda t)) \leqslant \lambda t w(x+c(\lambda)\, u(t)) \leqslant \lambda t w(c(\lambda)(x+u(t))) \leqslant c(\lambda)\, t w(x+u(t)) = c(\lambda)\, \psi(t,x)$. Making the substitution $t \to t/\lambda$ in the proved inequalities we obtain the second ones. ▮

Type 2. w is non-increasing and unbounded in $(0, d]$ and satisfies the inequality

$$w(x) \leqslant A_2 w(2x) \qquad \text{for every} \quad x \in (0, d/2]. \tag{3.4}$$

We define ψ again by (3.1). E.g., the weights $w(x) = x^\alpha |\log x|^\beta$ ($\alpha < 0$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$) are of this type. Now $\psi(t, \cdot)$ is non-increasing, u and v are strictly increasing, $v(0) = u(0) = 0$, and $w(x)$ tends to infinity when x tends to 0. The properties corresponding to these from Type 1 are

PROPERTY 2.1. *We have $v(\lambda x) \geqslant \lambda v(x)$ and $u(\lambda x) \leqslant \lambda u(x)$ for $\lambda > 1$.*

The proof is similar to the proof of Property 1.1.

PROPERTY 2.2. *Let $|x-y| \leqslant \lambda\psi(t,x)$. Then $\psi(t,x) \leqslant (A_2)^r\, \psi(t,y)$ for $r \geqslant \log_2(\lambda+1)$.*

Proof. We have $y < x + \lambda\psi(t,x) = x + \lambda t w(x+u(t)) \leqslant x + \lambda t w(u(t)) = x + \lambda u(t)$. Therefore $y + u(t) \leqslant 2^r(x+u(t))$ and (3.2) gives

$$\begin{aligned}\psi(t,x) = tw(x+u(t)) &\leqslant (A_2)^r t w(2^r(x+u(t))) \\ &\leqslant (A_2)^r t w(y+u(t)) = (A_2)^r \psi(t,y). \quad ▮\end{aligned}$$

PROPERTY 2.3. *For $\lambda > 1$ there is $\beta = \beta(A_2, \lambda) > 1$ such that*

$$\lambda\psi(t,x) \leqslant \psi(\beta\lambda t, x) \leqslant \lambda\beta\psi(t,x)$$

and

$$\psi(t,x)/(\beta\lambda) \leqslant \psi(t/(\beta\lambda), x) \leqslant \psi(t,x)/\lambda.$$

Proof. First we shall establish:

For every $\lambda > 1$ there exists $\alpha = \alpha(\lambda, A_2) > 1$ such that

$$\lambda u(t) \leqslant u(\alpha\lambda t). \tag{3.5}$$

From (3.4) we get $v(2x) = 2x/w(2x) \leqslant 2A_2 x/w(x) = 2A_2 v(x)$. Set $r = [\log_{2\lambda} + 1]$ and $\alpha = 2(A_2)^r$. Then $v(\lambda x) \leqslant v(2^r x) \leqslant (2A_2)^r v(x) \leqslant 2(\lambda_2)^r \lambda v(x) = \alpha\lambda v(x)$. Replacing x by $u(t)$ and using the fact that u is increasing we get (3.5).

Let $\beta = \alpha A_2$ where α is the constant from (3.5) for the multiplier λA_2, i.e.,

$\alpha=\alpha(\lambda A_2, A_2)>1$. If $x \geqslant u(\beta\lambda t)$ then (3.4) gives $\lambda\psi(t,x) = \lambda t w(x+u(t)) \leqslant \lambda t w(x) \leqslant A_2 \lambda t w(2x) \leqslant A_2 \lambda t w(x+u(\beta\lambda t)) \leqslant \psi(\beta\lambda t, x)/\alpha \leqslant \psi(\beta\lambda t, x)$. If $0<x\leqslant u(\beta\lambda t)$ then from (3.4) and (3.5) we obtain

$$\begin{aligned}\lambda\psi(t,x)&=\lambda t w(x+u(t))\leqslant \lambda t w(u(t))=\lambda u(t)\leqslant u(\beta\lambda t)/A_2\\&=\beta\lambda t w(u(\beta\lambda t))/A_2\leqslant \beta\lambda t w(2u(\beta\lambda t))\\&\leqslant \beta\lambda t w(x+u(\beta\lambda t))=\psi(\beta\lambda t,x).\end{aligned}$$

Moreover we have $x+u(t)\leqslant x+u(\lambda t)$ and $\lambda\psi(t,x)=\lambda t w(x+u(t))\geqslant \lambda t w(x+u(\lambda t))=\psi(\lambda t, x)$ and the first chain of inequalities is proved. The second chain is derived by the first one using the substitution $t\to t/(\beta\lambda)$. ∎

Type 3. v is non-increasing in $(0,d]$ and w satisfies the inequality $(t_0=v(d)/2)$

$$w(x)\leqslant A_3 w(x-t_0 w(x)) \qquad \text{for every} \quad x\in(0,d]. \tag{3.6}$$

E.g., the weights $w(x)=x^\alpha|\log x|^\beta$ ($\alpha>1$ and $\beta\in\mathbb{R}$ or $\alpha=1$ and $\beta\leqslant 0$) and the weights $w(x)=\exp(-x^{-\alpha})$ ($\alpha>0$) are of this type. Now w is strictly increasing, $w(0)=0$. We define ψ by

$$\psi(t,x)=tw(x). \tag{3.7}$$

PROPERTY 3.2. *Let* $0<t\leqslant t_0$, $0<\lambda\leqslant t_0/t$, $x, y\in(0,d]$, *and* $|x-y|\leqslant \lambda\psi(t,x)$. *Then* $\psi(t,x)\leqslant A_3\psi(t,y)$.

Proof. We have $y>x-\lambda t w(x)\geqslant x-t_0 w(x)$. Using (3.6) we get $\psi(t,x)\leqslant A_3 t w(x-t_0 w(x))\leqslant A_3\psi(t,y)$. ∎

In the neighbourhood $[d,\infty)$ of the end-point ∞ w will satisfy the following condition:

Type 4. w is monotone, v is non-decreasing in $[d,\infty)$, and in addition w satisfies the inequality $(t_0=v(d)/2)$

$$w(x)\leqslant A_4 w(x+t_0 w(x)) \qquad \text{for every} \quad x\in[d,\infty) \tag{3.8}$$

if w is decreasing.

For convenience we set $A_4=2$ if w is increasing.

E.g., the weights $w(x)=x^\alpha(\log x)^\beta$ ($\alpha<1$ and $\beta\in\mathbb{R}$ or $\alpha=1$ and $\beta\leqslant 0$) and the weights $w(x)=\exp(-x^\alpha)$ ($\alpha>0$) are of this type. We define ψ by (3.7).

PROPERTY 4.1. *For* $\lambda>1$ *we have* $w(\lambda x)\leqslant\lambda w(x)$.

Proof. $\lambda w(x)=\lambda x/v(x)\geqslant \lambda x/v(\lambda x)=w(\lambda x)$. ∎

PROPERTY 4.2. *Let* $0 < t \leqslant t_0$, $0 < \lambda \leqslant t_0/t$, $x, y \in [d, \infty)$, *and* $|x-y| \leqslant \lambda\psi(t, x)$. *Then* $\psi(t, x) \leqslant A_4 \psi(t, y)$.

Proof. Let w be increasing. Then $y \geqslant x - \lambda\psi(t, x) \geqslant x - t_0 w(x) = (2 - v(d)/v(x))x/2 \geqslant x/2$ and using Property 4.1 we get $\psi(t, x) = tw(x) \leqslant tw(2y) \leqslant 2tw(y) = 2\psi(t, y)$.

Let w be decreasing. Then $y \leqslant x + \lambda\psi(t, x) \leqslant x + t_0 w(x)$ and using (3.8) we get $w(x) \leqslant A_4 w(x + t_0 w(x)) \leqslant A_4 w(y)$. ∎

The weight w will satisfy the following global condition:

> There exist $A_5 \geqslant 1$, d_i, $a < d_3 < d_1 < d_2 < d_4 < b$, and weights w_1 in $[a, d_1]$ and w_2 in $[d_2, b]$ of some of the types described above such that $1/A_5 \leqslant w(x)/w_1(x) \leqslant A_5$ for $x \in [a, d_1]$, $1/A_5 \leqslant w(x)/w_2(x) \leqslant A_5$ for $x \in [d_2, b]$, and $1/A_5 \leqslant w(x) \leqslant A_5$ for $x \in [d_3, d_4]$. (3.9)

With v_j, u_j, and ψ_j we denote the functions associated with the weight w_j, $j = 1, 2$. Then we set

$$\underline{\psi}(t, x) = \begin{cases} (2k)^{-1}\psi_1(t, x) & \text{for } x \in [a, d_1]; \\ (2k)^{-1}\psi_2(t, x) & \text{for } x \in [d_2, b]; \\ \text{linear and continuous} & \text{in } [d_1, d_2]. \end{cases} \tag{3.10}$$

It follows from (3.10) that $\underline{\psi}(t, x)$ is equivalent to t for $x \in [d_3, d_4]$. The multiplier $(2k)^{-1}$ is chosen so that we shall be able to apply Property 1.2 in the next section. This multiplier is of importance only when w being of Type 1 does not satisfy (3.2). In the other cases we shall consider functions ψ equivalent to $\underline{\psi}$, i.e., ψ satisfying:

> There is $A > 1$ such that $1/A \leqslant \underline{\psi}(t, x)/\psi(t, x) \leqslant A$ for every $x \in [a, b]$ and the weights w_1 or w_2 from (3.9) satisfy (3.2) and (3.3) provided they are of Type 1. (3.11)

This condition will allow us to give an appropriate form to the argument of the τ modulus in (1.4) (see Corollaries 5.1, 5.2, 5.3, 5.4, and 5.5).

4. INEQUALITIES FOR MODULI OF DIFFERENTIABLE FUNCTIONS

First two theorems concern the usual moduli of smoothness. The author was not able to find any references on similar results. Applying these statements we derive a proof of the first inequality in (1.4).

THEOREM 4.1. *Let w be of Type* 1 *in* $[0, d]$ *satisfying* (3.2). *Then*

$$\omega_k(f; t)_p \leqslant c(k)\, A_1 v^k(t) \|w^k f^{(k)}\|_p.$$

Proof. From (2.9) and (2.8) we have

$$\begin{aligned}|\Delta_h^k f(x)| &\leqslant c(k) \int_0^{kh} y^{k-1}\, |f^{(k)}(x+y)|\, dy \\ &\leqslant c(k) \int_0^{kh} y^{k-1} w^{-k}(y)\, w^k(x+y)\, |f^{(k)}(x+y)|\, dy \\ &= c(k) \int_0^{kh} v^k(y)\, y^{-1} w^k(x+y)\, |f^{(k)}(x+y)|\, dy.\end{aligned}$$

Using this inequality, (3.2), and Property 1.1 we obtain

$$\begin{aligned}\|\Delta_h^k f(x)\|_p &\leqslant c(k) \int_0^{kh} v^k(y)\, y^{-1}\, \|w^k(\cdot + y)\, f^{(k)}(\cdot + y)\|_{p[0, d-kh]}\, dy \\ &\leqslant c(k) \int_0^{kh} v^k(y)\, y^{-1}\, dy\, \|w^k f^{(k)}\|_{p[0,d]} \\ &\leqslant c(k)\, A_1 v^k(kh) \|w^k f^{(k)}\|_p \leqslant c(k)\, A_1 v^k(h) \|w^k f^{(k)}\|_p.\end{aligned}$$

This proves the theorem because of the monotonicity of v. ∎

THEOREM 4.2. *For* $1 \leqslant p < \infty$, $0 < h \leqslant d$, *and* $f \in W^k_{p,\mathrm{loc}}[0, d]$ *we have*

$$\begin{aligned}&\|\Delta_h^k f(x)\|_{p[0, d-kh]} \\ &\quad\leqslant c(k)\, p \left\{ \int_0^h |y^k f^{(k)}(y)|^p\, dy + h^{kp} \int_h^{d-h} |f^{(k)}(y)|^p\, dy \right. \\ &\quad\quad \left. + \int_{d-h}^d |(d-y)^k f^{(k)}(y)|^p\, dy \right\}^{1/p}.\end{aligned}$$

Proof. From (2.9) and (2.8) we have

$$\begin{aligned}&|\Delta_h^k f(x)| \\ &\quad\leqslant c(k) \int_0^{kh} y^{k-1} |f^{(k)}(x+y)|\, dy \leqslant c(k) \int_x^{x+kh} y^{k-1} |f^{(k)}(y)|\, dy \\ &\quad\leqslant c(k) \left[\int_x^{x+kh} y^{kp-1+1/p'} |f^{(k)}(y)|^p\, dy \right]^{1/p} \left[\int_x^{x+kh} y^{-1-1/p}\, dy \right]^{1/p'} \\ &\quad\leqslant c(k) \left[\int_x^{x+kh} y^{kp-1/p}\, |f^{(k)}(y)|^p\, dy \right]^{1/p} p^{1/p'} x^{-1/(pp')}.\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^h |\Delta_h^k f(x)|^p\,dx
&\leqslant c(k)^p p^{p/p} \int_0^h x^{-1/p'} \int_x^{x+kh} y^{kp-1/p} |f^{(k)}(y)|^p\,dy\,dx \\
&\leqslant c(k)^p p^{p/p'} \int_0^{(k+1)h} y^{kp-1/p} |f^{(k)}(y)|^p \int_0^y x^{-1/p'}\,dx\,dy \\
&\leqslant c(k)^p p^p \int_0^{(k+1)h} y^{kp} |f^{(k)}(y)|^p\,dy \\
&\leqslant c(k)^p p^p \left[\int_0^h y^{kp} |f^{(k)}(y)|^p\,dy + h^{kp} \int_h^{(k+1)h} |f^{(k)}(y)|^p\,dy\right]. \qquad (4.1)
\end{aligned}$$

Changing the variables $x \to d-x$ and $h \to -h$ in (4.1) we obtain

$$\begin{aligned}
\int_{d-(k+1)h}^{d-kh} |\Delta_h^k f(x)|^p\,dx \leqslant c(k)^p p^p \Bigg[& h^{kp} \int_{d-(k+1)h}^{d-h} |f^{(k)}(y)|^p\,dy \\
&+ \int_{d-h}^{d} (d-y)^{kp} |f^{(k)}(y)|^p\,dy \Bigg]. \qquad (4.2)
\end{aligned}$$

Moreover we have

$$\begin{aligned}
&\int_h^{d-(k+1)h} |\Delta_h^k f(x)|^p\,dx \\
&\quad \leqslant \omega_k(f;h)^p_{p[h,d-h]} \leqslant h^{kp} \|f^{(k)}\|^p_{p[h,d-h]} = h^{kp} \int_h^{d-h} |f^{(k)}(y)|^p\,dy. \qquad (4.3)
\end{aligned}$$

Combining (4.1), (4.2), and (4.3) we obtain the statement of the theorem. ∎

COROLLARY 4.1. *If $1 \leqslant p < \infty$ and w is of Type 1 in $[0, d]$ or w is symmetric in $[0, d]$ and is of Type 1 in $[0, d/2]$ then*

$$\omega_k(f;t)_p \leqslant c(k)\, p v^k(t) \|w^k f^{(k)}\|_p.$$

In comparison with Theorem 4.1 we do not require w to satisfy (3.2) in Corollary 4.1 but we have to pay for this by excluding the case $p = \infty$.

LEMMA 4.1. *Let w be of Type i, $i = 1, 2, 3, 4$, and let w satisfy* (3.2) *being* *of Type* 1, $0<t\leqslant c(w)$, $\eta=1/(2k)$, $[\alpha,\beta]=[0,d]$ *if* w *is not of Type* 4 *and* $[\alpha,\beta]=[d,\infty)$ *if* w *is of Type* 4. *Then*

$$\tau_k(g;\eta\psi(t))_{p,p[\alpha,\beta]}\leqslant c(k,w)\,\|g\|_{p[\alpha,\beta]}\qquad \text{if}\quad g\in L_p(\alpha,\beta] \tag{4.4}$$

$$\tau_k(g;\eta\psi(t))_{p,p[\alpha,\beta]}\leqslant c(k,w)\,t^k\,\|w^k g^{(k)}\|_{p[\alpha,\beta]}\qquad \text{if}\quad g\in W_p^k(w). \tag{4.5}$$

Moreover the statement of the lemma is true for $p<\infty$ *and arbitrary weights of Type* 1 *but with constant* $c(k,w,p)$ *in* (4.5).

Proof. From Properties i.2 ($i=1,2,3,4$) we get $|x-y|\leqslant c(w)\,\psi(t,y)$ provided $x,y\in[\alpha,\beta]$ and $|x-y|\leqslant\eta k\psi(t,x)$. Therefore for every $y\in[\alpha,\beta]$ we have

$$\operatorname{meas}\{x\in[\alpha,\beta]:|x-y|\leqslant\eta k\psi(t,x)\}\leqslant c(w)\,\psi(t,y). \tag{4.6}$$

From (2.6), Properties i.2, and (4.6) we obtain

$$\begin{aligned}
&\tau_k(g;\eta\psi(t))_{p,p[\alpha,\beta]}\\
&=\left[\int_\alpha^\beta \omega_k(g,x;\eta\psi(t,x))_p^p\,dx\right]^{1/p}\\
&\leqslant\left[\int_\alpha^\beta |g(x)|^p\,dx\right]^{1/p}\\
&\qquad+c(k)\left[\int_\alpha^\beta 2/\psi(t,x)\int_{-\eta k\psi(t,x)}^{\eta k\psi(t,x)}|g(x+z)|^p\,dz\,dx\right]^{1/p}\\
&\leqslant\|g\|_p+c(k,w)\left[\int_\alpha^\beta\int_{-\eta k\psi(t,x)}^{\eta k\psi(t,x)}|g(x+z)|^p/\psi(t,x+z)\,dz\,dx\right]^{1/p}\\
&\leqslant\|g\|_p+c(k,w)\left[\int_\alpha^\beta|g(y)|^p\,dy\right]^{1/p}=c(k,w)\,\|g\|_p
\end{aligned}$$

which proves (4.4).

Reasoning as above and using (2.7) instead of (2.6) we get

$$\begin{aligned}
&\tau_k(g;\eta\psi(t))_{p,p[\alpha,\beta]}\\
&\qquad\leqslant c(k)\left[\int_\alpha^\beta\psi^{kp-1}(t,x)\int_{-\eta k\psi(t,x)}^{\eta k\psi(t,x)}|g(x+z)|^p\,dz\,dx\right]^{1/p}\\
&\qquad\leqslant c(k,w)\left[\int_\alpha^\beta\psi^{kp}(t,y)\,|g^{(k)}(y)|^p\,dy\right]^{1/p}.
\end{aligned}$$

From this inequality we obtain (4.4) when w is not of Type 1 because

$\psi(t, y) \leqslant tw(y)$ in this case. Let w be of Type 1. We divide $[0, d]$ into two parts:

$$\tau_k(g; \eta\psi(t))_{p,p[0,d]} \leqslant \left[\int_0^{4u(t)} \omega_k(g, x; \eta\psi(t, x))_p^p \, dx\right]^{1/p} + \left[\int_{4u(t)}^{d} \omega_k(g, x; \eta\psi(t, x))_p^p \, dx\right]^{1/p}. \tag{4.7}$$

Let $x \in [4u(t), d]$ and $|x-y| \leqslant \eta k\psi(t, x)$. From Property 1.2.b we have $y \geqslant x/4 \geqslant u(t)$ and from Property 1.1 we get $\psi(t, x) \leqslant tw(4y) \leqslant 4tw(y)$. Using (2.7), Property 1.2.a, and (4.6) we obtain

$$\begin{aligned}
&\left[\int_{4u(t)}^{d} \omega_k(g, x; \eta\psi(t, x))_p^p \, dx\right]^{1/p} \\
&\quad \leqslant c(k)\left[\int_{4u(t)}^{d} \psi^{kp-1}(t, x) \int_{-\eta k\psi(t,x)}^{\eta k\psi(t,x)} |g^{(k)}(x+z)|^p \, dz \, dx\right]^{1/p} \\
&\quad \leqslant c(k, w)\left[\int_{u(t)}^{d} |g^{(k)}(y)|^p \, \psi^{kp}(t, y) \, dy\right]^{1/p} \\
&\quad \leqslant c(k) \, t^k \, \|w^k g^{(k)}\|_{p[u(t),d]}.
\end{aligned} \tag{4.8}$$

For $x \in [0, 4u(t)]$ from Property 1.1 we get $u(t) \leqslant \psi(t, x) \leqslant tw(5u(t)) < 5tw(u(t)) = 5u(t)$. Now (2.5) and Lemma 2.3 give

$$\begin{aligned}
&\left[\int_0^{4u(t)} \omega_k(g, x; \eta\psi(t, x))_p^p \, dx\right]^{1/p} \\
&\quad \leqslant 5\left[\int_0^{4u(t)} \omega_k(g, x; 5\eta u(t))_p^p \, dx\right]^{1/p} \\
&\quad \leqslant 5\tau_k(g; 5\eta u(t))_{p,p[0,d]} \leqslant c(k) \, \omega_k(g; u(t))_{p[0,d]}.
\end{aligned} \tag{4.9}$$

Now (4.5) follows from (4.7), (4.8), (4.9), and Theorem 4.1 if w satisfies (3.2) or Corollary 4.1 otherwise. ∎

Now we are ready to prove the first inequality in (1.4).

THEOREM 4.3. *Let w satisfy* (3.9) *in* $[a, b]$, *let the weights w_1 and w_2 satisfy* (3.2) *being of Type* 1, *and let ψ satisfy* (3.10) *for* $0 < t \leqslant c(w)$. *Then for every $f \in L_p[a, b] + W_p^k(w)$ we have*

$$c(k, w) \, \tau_k(f; \psi(t))_{p,p[a,b]} \leqslant K(t^k, f; L_p[a, b], W_p^k(w)). \tag{4.10}$$

If w_1 or w_2 does not satisfy (3.2) *being of Type* 1 *then* (4.10) *is true for $p < \infty$ with a constant depending also on p.*

Proof. From (3.10) and Lemma 2.3 we get $(\eta = 1/(2k))$

$$\begin{aligned}
&\tau_k(f; \psi(t))_{p, p(D)} \\
&\quad \leqslant \left[\int_a^{d_3} \omega_k(f, x; \psi(t, x))_p^p \, dx\right]^{1/p} \\
&\qquad + \left[\int_{d_3}^{d_4} \omega_k(f, x; \psi(t, x))_p^p \, dx\right]^{1/p} + \left[\int_{d_4}^{b} \omega_k(f, x; \psi(t, x))_p^p \, dx\right]^{1/p} \\
&\quad \leqslant c(k, w)\{\tau_k(f; \eta\psi_1(t))_{p, p[a, d_3]} + \omega_k(f; t)_{p[d_3, d_4]} \\
&\qquad + \tau_k(f; \eta\psi_2(t))_{p, p[d_4, b]}\}.
\end{aligned} \tag{4.11}$$

From (2.2) and Lemma 4.1 for any $g \in W_p^k(w)$ we have

$$\begin{aligned}
&\tau_k(f; \eta\psi_1(t))_{p, p[a, d_3]} \\
&\quad \leqslant \tau_k(f - g; \eta\psi_1(t))_{p, p[a, d_3]} + \tau_k(g; \eta\psi_1(t))_{p, p[a, d_3]} \\
&\quad \leqslant c(k, w)\{\|f - g\|_{p[a, b]} + t^k \|w^k g^{(k)}\|_{p[a, b]}\}.
\end{aligned}$$

Therefore

$$\tau_k(f; \eta\psi_1(t))_{p, p[a, d_3]} \leqslant c(k, w)\, K(t^k, f; L_p, W_p^k(w)). \tag{4.12}$$

In the same manner we get

$$\tau_k(f; \eta\psi_2(t))_{p, p[d_4, b]} \leqslant c(k, w)\, K(t^k, f; L_p, W_p^k(w)). \tag{4.13}$$

From (1.3) and (3.9) we obtain

$$\begin{aligned}
\omega_k(f; t)_{p[d_3, d_4]} &\leqslant c(k)\, K(t^k, f; L_p[d_3, d_4], W_p^k(1)) \\
&\leqslant c(k, w)\, K(t^k, f; L_p[a, b], W_p^k(w)).
\end{aligned} \tag{4.14}$$

Combining (4.11), (4.12), (4.13), and (4.14) we prove (4.10). ∎

The next example shows that (4.10) is not valid when $p = \infty$ and that w is an arbitrary weight of Type 1.

Example. *Let* $w(x) = x(1 - \ln x)^{1/(2k)}$, $f(x) = (1 - \ln x)^{1/2}$ for $x \in [0, 1]$. Then $\tau_k(f; \psi(t))_{\infty, \infty[0,1]} = \infty$ because f is monotone and unbounded but $\|w^k f^{(k)}\|_\infty \leqslant c(k)$ and therefore $K(t^k, f; C, W_\infty^k(w)) = O(t^k)$.

5. A Characterization of the Weighted K-Functional

In this section we prove the second inequality in (1.4) and complete the characterization of the weighted K-functional. In order to do this we first construct appropriate intermediate functions when w is of different types.

Lemma 5.1. *Let w be a weight of Type 1 in $[0, d]$, $0<t\leqslant v(d)$ and $0<\eta\leqslant 1/(2k)$. Then for every $f\in L_p[0, d]$ there exists $g\in W_p^k(1)$ such that $g^{(k)}(x)=0$ for $x\in[0, \eta k u(t)/3]$,*

$$\|f-g\|_{p[0,d]}\leqslant c(k)\,\tau_k(f;\eta\psi(t))_{1,p[0,d]}, \tag{5.1}$$

$$\|(\eta k\psi(t))^k g^{(k)}\|_{p[0,d]}\leqslant c(k)\,\tau_k(f;\eta\psi(t))_{1,p[0,d]}. \tag{5.2}$$

Proof. We set $y_0=0$, $y_{j+1}=y_j+h_j$, $h_j=\eta k\psi(t, y_j)/3$, $j=0, 1, \ldots$. There exists n such that $y_n<d\leqslant y_{n+1}$ because of $h_j\geqslant \eta k u(t)/3>0$. We shall work only with the points $y_0, y_1, \ldots, y_{n+1}$ where we set $y_{n+1}=d$. From the monotonicity of $\psi(t,\cdot)$ and Property 1.2 we get

$$\begin{aligned} &1\leqslant h_{j+1}/h_j\leqslant 2,\\ &1\leqslant \eta k\psi(t,x)/(3h_j)\leqslant 2 \qquad \text{for every}\quad x\in[y_j, y_{j+1}]. \end{aligned} \tag{5.3}$$

Let μ be the function from Section 2. We set $\mu_0(x)=1-\mu((x-y_0)/h_1)$, $\mu_j(x)=\mu((x-y_j)/h_j)[1-\mu((x-y_{j+1})/h_{j+1})]$ for $j=1, 2, \ldots, n-2$ and $\mu_{n-1}(x)=\mu((x-y_{n-1})/h_{n-1})$. Then $\sum_{j=0}^{n-1}\mu(x)=1$ for every $x\in[0, d]$ and the only functions μ_j which do not vanish in $[y_j, y_{j+1}]$ are μ_0 for $j=0$, μ_{j-1} and μ_j for $j=1, 2, \ldots, n-1$, and μ_{n-1} for $j=n$.

We denote by $Q_j\in H_{k-1}$ the polynomial of best algebraic L_p approximation of degree $k-1$ to f in the interval $[y_j, y_{j+2}]$, $j=0, 1, \ldots, n-1$. From (2.1), (5.3), Lemma 2.3, and (2.5) we get

$$\begin{aligned} \|f-Q_j\|_{p[y_j,y_{j+2}]} &\leqslant c(k)\,\omega_k(f;(h_j+h_{j+1})/k)_{p[y_j,y_{j+2}]}\\ &\leqslant c(k)\,\omega_k(f;3h_j/k)_{p[y_j,y_{j+2}]}\leqslant c(k)\,\tau_k(f;3h_j/k)_{1,p[y_j,y_{j+2}]}\\ &\leqslant c(k)\,\tau_k(f;\eta\psi(t))_{1,p[y_j,y_{j+2}]}. \end{aligned} \tag{5.4}$$

We set

$$g(x)=\sum_{j=0}^{n-1}\mu_j(x)\,Q_j(x). \tag{5.5}$$

Now from (5.5) and (5.4) we get

$$\begin{aligned}\|f-g\|_{p[0,d]}^p &= \left\| \sum_{j=0}^{n-1} \mu_j(f-Q) \right\|_{p[0,d]}^p \\ &\leqslant \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+2}} |f(x)-Q_j(x)|^p \, dx \\ &\leqslant c(k)^p \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+2}} \omega_k(f, x; \eta\psi(t, x))_1^p \, dx \\ &\leqslant c(k)^p \, \tau_k(f; \eta\psi(t))_{1, p[0,d]}^p . \end{aligned} \tag{5.6}$$

So we have proved (5.1). It follows from (5.5) that $g \in C^\infty[0, d]$ and $g^{(k)}(x) = 0$ for $x \in [y_0, y_1] \cup [y_n, y_{n+1}]$. Let $x \in [y_j, y_{j+1}]$ for some $j = 1, 2, ..., n-1$. Then $g(x) = Q_{j-1}(x) + \mu((x - y_j)/h_j) \, \phi(x)$ where $\phi = Q_j - Q_{j-1}$ and therefore

$$\|h_j^k g^{(k)}\|_{p[y_j, y_{j+1}]} \leqslant \sum_{r=0}^{k} \binom{k}{r} \|\mu^{(k-r)}\|_\infty \, h_j^r \, \|\phi^{(r)}\|_{p[y_j, y_{j+1}]}. \tag{5.7}$$

From Lemma 2.4 (or Markov's inequality) and (5.4) we obtain

$$\begin{aligned} h_j^r \|\phi^{(r)}\|_{p[y_j, y_{j+1}]} &\leqslant c(k) \, \|\phi\|_{p[y_j, y_{j+1}]} \\ &\leqslant c(k) \{ \|Q_j - f\|_{p[y_j, y_{j+1}]} + \|Q_{j-1} - f\|_{p[y_j, y_{j+1}]} \} \\ &\leqslant c(k) \, \tau_k(f; \eta\psi(t))_{1, p[y_{j-1}, y_{j+2}]}. \end{aligned} \tag{5.8}$$

It follows from (5.7) and (5.8) that

$$\|h_j^k \, g^{(k)}\|_{p[y_j, y_{j+1}]} \leqslant c(k) \, \tau_k(f; \eta\psi(t))_{1, p[y_{j-1}, y_{j+2}]}. \tag{5.9}$$

Using (5.3) and (5.9) we get

$$\begin{aligned} &\|(\eta k\psi(t))^k g^{(k)}\|_{p[0,d]}^p \\ &\quad = \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+1}} |(\eta k\psi(t, x))^k g^{(k)}(x)|^p \, dx \\ &\quad \leqslant 6^{kp} \sum_{j=0}^{n-1} h_j^{kp} \int_{y_j}^{y_{j+1}} |g^{(k)}(x)|^p \, dx \\ &\quad \leqslant c(k)^p \sum_{j=0}^{n-1} \int_{y_{j-1}}^{y_{j+2}} \omega_k(f, x; \eta\psi(t, x))_1^p \, dx \\ &\quad \leqslant c(k)^p \tau_k(f; \eta\psi(t))_{1, p[0,d]}^p. \end{aligned}$$

This completes the proof. ∎

LEMMA 5.2. *Let w be a weight of Type 2 in $[0, d]$, $0 < t \leqslant v(d)$, let A_2 be the constant from (3.4), and let $0 < \eta < 1/k$. Then for every $f \in L_p[0, d]$ there exists $g \in W_p^k(1)$ such that $g^{(k)}(x) = 0$ for $x \in [0, \eta k u(t)/(1 + A_2)]$,*

$$\|f - g\|_{p[0,d]} \leqslant c(k, A_2)\, \tau_k(f; \eta\psi(t))_{1,p[0,d]},$$

and

$$\|(\eta k \psi(t)^k g^{(k)}\|_{p[0,d]} \leqslant c(k, A_2)\, \tau_k(f; \eta\psi(t))_{1,p[0,d]}.$$

Proof. We set $y_0 = 0$, $y_{j+1} = y_j + h_j$, $h_j = \eta k \psi(t, y_j)/(A_2 + 1)$, $j = 0, 1, \ldots$. From Property 2.2 we get

$$1 \leqslant h_j/h_{j+1} \leqslant A_2 \qquad \text{and} \qquad 1 \leqslant (1 + A_2)\, h_j/(\eta k \psi(t, x)) \leqslant A_2$$

for every $x \in [y_j, y_{j+1}]$. Now we proceed as in the proof of Lemma 5.1. ∎

LEMMA 5.3. *Let w be a weight of Type 3 in $[0, d]$, $0 < t \leqslant t_0 = v(d)/2$, let A_3 be the constant from (3.6), and let $0 < \eta \leqslant t_0/(kt)$. Then for every $f \in L_p[0, d] + W_p^k(w)$ there exists $g \in W_p^k(w)$ such that*

$$\|f - g\|_{p[0,d]} \leqslant c(k, A_3)\, \tau_k(f; \eta\psi(t))_{1,p[0,d]}$$

and

$$\|(\eta k \psi(t))^k g^{(k)}\|_{p[0,d]} \leqslant c(k, A_3)\, \tau_k(f; \eta\psi(t))_{1,p[0,d]}.$$

Proof. We set $y_0 = d$, $y_{j+1} = y_j - h_j$, $h_j = \eta k \psi(t, y_j)/(1 + A_3) = \eta k t w(y_j)/(1 + A_3)$. In this case we have $h_j \leqslant t_0 w(y_j) = v(d)\, y_j/(2v(y_j)) \leqslant y_j/2$ and therefore y_j is well defined for every natural j. Let $y \in (0, d)$ and $j > (d - y)(1 + A_2)/(\eta k t w(y))$. Then $y_j < y$ and hence $\lim y_j = 0$ when $j \to \infty$. From Property 3.2 we have

$$1 \leqslant h_j/h_{j+1} \leqslant A_3 \qquad \text{and} \qquad 1 \leqslant (1 + A_3)\, h_j/(\eta k \psi(t, x)) \leqslant A_3$$

for every $x \in [y_{j+1}, y_j]$. Now we proceed as in the proof of Lemma 5.1 but the summation is to infinity. ∎

LEMMA 5.4. *Let w be a weight of Type 4 in $[d, \infty)$, $0 < t \leqslant t_0 = v(d)/2$, let A_4 be the constant from (3.8), and let $0 < \eta \leqslant t_0/(kt)$. Then for every $f \in L_p[d, \infty) + W_p^k(w)$ there exists $g \in W_p^k(w)$ such that*

$$\|f - g\|_{p[d,\infty)} \leqslant c(k, A_4)\, \tau_k(f; \eta\psi(t))_{1,p[d,\infty)}$$

and

$$\|(\eta k \psi(t))^k g^{(k)}\|_{p[d,\infty)} \leqslant c(k, A_4)\, \tau_k(f; \eta\psi(t))_{1,p[d,\infty)}.$$

Proof. We set $y_0 = d$, $y_{j+1} = y_j + h_j$, $h_j = \eta k \psi(t, y_j)/(1 + A_4)$. As in the proof of Lemma 5.3 we get $y_j \to \infty$ when $j \to \infty$. From Property 4.2 we obtain

$$1 \leqslant (h_j/h_{j+1})^{\varepsilon} \leqslant A_4 \qquad \text{and} \qquad 1 \leqslant ((1 + A_3)\, h_j/(\eta k \psi(t, x)))^{\varepsilon} \leqslant A_4$$

for every $x \in [y_j, y_{j+1}]$, where $\varepsilon = 1$ if w is non-increasing and $\varepsilon = -1$ if w is non-decreasing. Now we proceed as in the proof of Lemma 5.1. ∎

Let us remark that all four lemmas can be applied for $f \in L_p + W_p^k(w)$ because $L_p + W_p^k(w) = L_p$ when w is of Type 1 or 2.

THEOREM 5.1. *Let w satisfy* (3.9), *let ψ be given by* (3.10), *and let $0 < t \leqslant c(w)$. Then for every $f \in L_p[a, b] + W_p^k(w)$ there is $g \in W_p^k(w)$ such that $g^{(k)}(x) = 0$ in a neighbourhood of the left end-point a with length $c(w)\, u_1(t)$ provided w_1 is of Types* 1 *or* 2, *$g^{(k)}(x) = 0$ in a neighbourhood of the right end-point b with length $c(w)\, u_2(t)$ provided w_2 is of Types* 1 *or* 2, *and*

$$\|f - g\|_{p[a,b]} + t^k \|w^k g^{(k)}\|_{p[a,b]} \leqslant c(k, w)\, \tau_k(f; \psi(t))_{1, p[a,b]}.$$

Proof. We shall use Lemmas 5.i, $i = 1, 2, 3, 4$, with $\eta = 1/(2k)$. Let g_1 and g_2 be the functions from Lemmas 5.i for the weights w_1 and w_2 in the intervals $[a, d_1]$ and $[d_2, b]$, respectively. Let g_3 be the function from Lemma 5.1 for the weight $w = 1$ in the interval $[d_3, d_4]$ (we can also use the modify Steklov function for f in this interval as g_3). We set

$$\begin{aligned} g(x) = {} & \mu((x - d_3)/(d_1 - d_3))[1 - \mu((x - d_2)/(d_4 - d_2))]\, g_3(x) \\ & + [1 - \mu((x - d_3)/(d_1 - d_3))]\, g_1(x) + \mu((x - d_2)/(d_4 - d_2))\, g_2(x). \end{aligned} \tag{5.10}$$

From Lemmas 5.i and (5.10) we get

$$\begin{aligned} \|f - g\|_{p[a,b]} & \leqslant \|f - g_1\|_{p[a,d_1]} + \|f - g_3\|_{p[d_3,d_4]} + \|f - g_2\|_{p[d_2,b]} \\ & \leqslant c(k, w)\, \tau_k(f; \psi(t))_{1, p[a,b]} \end{aligned}$$

and $g^{(k)}$ vanishes in suitable neighbourhoods of the end-points. Therefore

$$t^k \|w^k g^{(k)}\|_p \leqslant c(k, w)\, \|\psi(t)^k g^{(k)}\|_p. \tag{5.11}$$

Indeed $tw(x) \leqslant \psi(t, x)$ when w is not of Type 2 and for w of Type 2 (in $[0, d]$) and $x > \lambda u(t)$ using (3.4) we get

$$tw(x) \leqslant c(w, \lambda)\, tw(x + x/\lambda) \leqslant c(w, \lambda)\, tw(x + u(t)) \leqslant c(w, \lambda)\, \psi(t, x).$$

From (5.10), Lemma 2.4, (3.10), (5.11), and Lemmas 5.i we get

$$
\begin{aligned}
&\|w^k g^{(k)}\|_{p[a,b]} \\
&\leqslant c(k, w)\left\{\|w_1^k g_1^{(k)}\|_{p[a,d_1]} + \|w_2^k g_2^{(k)}\|_{p[d_2,b]} + \|g^{(k)}\|_{p[d_3,d_4]}\right. \\
&\quad \left. + \sum_{j=0}^{k} \|g_1^{(j)} - g_3^{(j)}\|_{p[d_3,d_1]} + \sum_{j=0}^{k} \|g_3^{(j)} - g_2^{(j)}\|_{p[d_4,d_2]}\right\} \\
&\leqslant c(k, w)\{\|w_1^k g_1^{(k)}\|_{p[a,d_1]} + \|w_2^k g_2^{(k)}\|_{p[d_2,b]} + \|g_3^{(k)}\|_{p[d_3,d_4]} \\
&\quad + \|g_1 - f\|_{p[d_3,d_1]} + \|g_3 - f\|_{p[d_3,d_4]} + \|g_2 - f\|_{p[d_2,d_4]}\} \\
&\leqslant c(k, w)\, t^{-k} \tau_k(f; \psi(t))_{1,p[a,b]}
\end{aligned}
$$

which proves the theorem. ∎

Combining Theorem 3.4, (1.1), and Theorem 5.1 we obtain

THEOREM 5.2. *Let w satisfy (3.9) in $[a, b]$, let the weights w_1 and w_2 satisfy (3.2) being of Type 1, and let $\underline{\psi}$ satisfy (3.10) for $0 < t \leqslant c(w)$. Then for every $f \in L_p[a, b] + W_p^k(w)$ we have*

$$
\begin{aligned}
c(k, w)\, \tau_k(f; \underline{\psi}(t))_{p,p[a,b]} &\leqslant K(t^k, f; L_p[a, b], W_p^k(w)) \\
&\leqslant c(k, w)\, \tau_k(f; \underline{\psi}(t))_{1,p[a,b]}.
\end{aligned}
$$

If w_1 or w_2 does not satisfy (3.2) being of Type 1 then the first inequality above is true for $p < \infty$ with constant depending also on p.

In order to allow more flexibility in the choice of the argument function $\underline{\psi}$ of τ moduli in the above theorem we shall prove

THEOREM 5.3. *Let w satisfy (3.9) in $[a, b]$ and let ψ satisfy (3.11) for $0 < t \leqslant c(w)$. Then for every $f \in L_p[a, b] + W_p^k(w)$ we have*

$$
\begin{aligned}
c(k, w)\, \tau_k(f; \psi(t))_{p,p[a,b]} &\leqslant K(t^k, f; L_p[a, b], W_p^k(w)) \\
&\leqslant c(k, w)\, \tau_k(f; \psi(t))_{1,p[a,b]}. \qquad (5.12)
\end{aligned}
$$

Proof. In view of the inequality

$$
\begin{aligned}
\tau_k(f; \psi(t))_{p,p[a,b]} &\leqslant \tau_k(f; \psi(t))_{p,p[a,d_3]} \\
&\quad + \tau_k(f; \psi(t))_{p,p[d_3,d_4]} + \tau_k(f; \psi(t))_{p,p[d_4,b]} \qquad (5.13)
\end{aligned}
$$

for proving the first inequality in (5.12) it is enough to estimate every term in the right-hand side of (5.13) with the K-functional in (5.12). From (3.11), (2.5), Lemma 2.3, and (1.3) we get

$$\begin{aligned}
&\tau_k(f; \psi(t))_{p,p[d_3,d_4]}\\
&\quad\leqslant c(k,w)\,\tau_k(f; c(k,w)t)_{p,p[d_3,d_4]}\\
&\quad\leqslant c(k,w)\,\omega_k(f; c(k,w)t)_{p[d_3,d_4]} \leqslant c(k,w)\,\omega_k(f;t)_{p[d_3,d_4]}\\
&\quad\leqslant c(k,w)\,K(t^k, f; L_p[d_3,d_4], W_p^k(1)) \leqslant c(k,w)\,K(t^k, f; L_p[a,b], W_p^k(w)).
\end{aligned}$$

In view of the symmetry of $[a, d_3]$ and $[d_4, b]$ we shall only consider the modulus in the first interval. From (3.11) we have $\psi(t,x) \leqslant A\psi_1(t,x)/(2k) \leqslant A^2\psi(t,x)$ and now (2.5) gives

$$\tau_k(f; \psi(t))_{p,p[a,d_3]} \leqslant A_2\tau_k(f; A\psi_1(t)/(2k))_{p,p[a,d_3]}. \tag{5.14}$$

If w_1 is of Type 3 or 4 then $A\psi_1(t) = \psi_1(At)$. If w_1 is of Type 1 or 2 then Property 1.3 or 2.3 with $\lambda = A$ gives ($\beta = 1$ for w_1 of Type 1) $A\psi_1(t,x)/(2k) \leqslant \psi_1(\beta At, x)/(2k) \leqslant c(A)\,\psi_1(t,x)/(2k)$. Now (2.5) gives

$$\begin{aligned}
&\tau_k(f; A\psi_1(t)/(2k))_{p,p[a,d_3]}\\
&\qquad\leqslant c(A)\,\tau_k(f; (\psi_1(\beta At)/(2k))_{p,p[a,d_3]}.
\end{aligned} \tag{5.15}$$

Now from (5.14), (5.15), Theorem 4.3, and (1.1) we get

$$\begin{aligned}
\tau_k(f; \psi(t))_{p,p[a,d_3]} &\leqslant c(A)\,\tau_k(f; \psi_1(\beta At)/(2k))_{p,p[a,d_3]}\\
&\leqslant c(k,w)\,K((\beta At)^k, f; L_p[a,d_3], W_p^k(w))\\
&\leqslant c(k,w)\,K(t^k, f; L_p[a,b], W_p^k(w))
\end{aligned}$$

which completes the proof of the first inequality in (5.12). For the proof of the second inequality in (5.12) we use the same statements arguing in reverse order. ∎

Now we shall derive corollaries from Theorem 5.3 for the weights mentioned in Section 1.

Let $w(x) = \sqrt{(x - x^2)}$, $x \in [0, 1]$. We can choose $d_1 = \frac{1}{3}$, $d_2 = \frac{2}{3}$, $d_3 = \frac{1}{4}$, $d_4 = \frac{3}{4}$, $w_1(x) = \sqrt{x}$, and $w_2(x) = \sqrt{(1-x)}$. Then $u_1(t) = t^2$ and $\psi_1(t,x) = t\sqrt{(x + t^2)}$. Therefore we can choose $\psi(t,x) = tw(x) + t^2$.

Corollary 5.1. *For $\psi(t,x) = t\sqrt{(x - x^2)} + t^2$ we have*

$$\begin{aligned}
c(k,w)\,\tau_k(f; \psi(t))_{p,p[0,1]} &\leqslant K(t^k, f; L_p[0,1], W_p^k(\sqrt{(x-x^2)}))\\
&\leqslant c(k,w)\,\tau_k(f; \psi(t))_{p,p[0,1]}.
\end{aligned}$$

Reasoning in the same manner we get

COROLLARY 5.2. *For $\psi(t, x) = t\sqrt{(1-x^2)} + t^2$ we have*

$$c(k, w)\, \tau_k(f; \psi(t))_{p, p[-1,1]} \leqslant K(t^k, f; L_p[-1, 1], W_p^k(\sqrt{(1-x^2)}))$$
$$\leqslant c(k, w)\, \tau_k(f; \psi(t))_{p, p[-1,1]}.$$

COROLLARY 5.3. *For $\psi(t, x) = t\sqrt{x} + t^2$ we have*

$$c(k, w)\, \tau_k(f; \psi(t))_{p, p[0,\infty)} \leqslant K(t^k, f; L_p[0, \infty), W_p^k(\sqrt{x}))$$
$$\leqslant c(k, w)\, \tau_k(f; \psi(t))_{p, p[0,\infty)}.$$

COROLLARY 5.4. *For $\psi(t, x) = t\sqrt{(x+x^2)} + t^2$ we have*

$$c(k, w)\, \tau_k(f; \psi(t))_{p, p[0,\infty)} \leqslant K(t^k, f; L_p[0, \infty), W_p^k(\sqrt{(x+x^2)}))$$
$$\leqslant c(k, w)\, \tau_k(f; \psi(t))_{p, p[0,\infty)}.$$

At the end we shall derive a result having applications in best approximation by algebraic polynomials and approximation by operators. Let w be symmetry in $[0, 1]$ (i.e., $w(1-x) = w(x)$) or let w be a weight in $[0, \infty)$ and in both cases let w_1 from (3.9) be of Type 1 in $[0, \frac{1}{3}]$ satisfying (3.2). Let us denote by u the function u_1 corresponding to w_1. We set

$$K^*(t^k, f; L_p, W_p^k(w)) = \inf\{\|f-g\|_p + t^k \|w^k g^{(k)}\|_p + u(t)^k \|g^{(k)}\|_p\}.$$

THEOREM 5.4. *Under the above assumption we have*

$$K(t^k, f) \leqslant K^*(t^k, f) \leqslant c(k, w)\, K(t^k, f). \tag{5.16}$$

Proof. The first inequality in (5.16) follows directly from the definitions. Let g be the function from Theorem 5.1 corresponding to f. We have

$$u(t)^k \|g^{(k)}\|_p \leqslant c(k, w)\, t^k \|w^k g^{(k)}\|_p$$

because $g^{(k)} = 0$ in $[0, c(w)u(t)]$. Combining this inequality with Theorems 5.1 and 5.2 and (2.4) we get

$$\begin{aligned} K^*(t^k, f) &\leqslant \|f-g\|_p + t^k \|w^k g^{(k)}\|_p + u(t)^k \|g^{(k)}\|_p \\ &\leqslant c(k, w)\{\|f-g\|_p + t^k \|w^k g^{(k)}\|_p\} \\ &\leqslant c(k, w)\, \tau_k(f; \underline{\psi}(t))_{1,p} \\ &\leqslant c(k, w)\, K(t^k, f). \quad \blacksquare \end{aligned}$$

As a corollary of the last theorem one can mention the inequalities

$$\begin{aligned}\inf\{\|f-g\|_{p[0,1]} + t^k \|(x-x^2)^{k/2} g^{(k)}(x)\|_{p[0,1]} \\ + t^{2k} \|g^{(k)}\|_{p[0,1]} : g\} \\ \leqslant c(k) \inf\{\|f-g\|_{p(0,1]} + t^k \|(x-x^2)^{k/2} g^{(k)}(x)\|_{p[0,1]} : g\}\end{aligned}$$

and

$$\begin{aligned}\inf\{\|f-g\|_{p[0,\infty)} + t^k \|x^{k/2} g^{(k)}(x)\|_{p[0,\infty)} \\ + t^{2k} \|g^{(k)}\|_{p[0,\infty)} : g\} \\ \leqslant c(k) \inf\{\|f-g\|_{p[0,\infty)} + t^k \|x^{k/2} g^{(k)}(x)\|_{p[0,\infty)} : g\}.\end{aligned}$$

6. Properties of the Moduli

In this section we derive several properties of the moduli following from their equivalence with the weighted K-functional.

THEOREM 6.1. *Let ψ satisfy* (3.11) *for some w satisfying* (3.9), $0 < t \leqslant c(w)$ *and* $1 \leqslant q \leqslant r \leqslant p$. *Then*

$$\tau_k(f; \psi(t))_{q,p} \leqslant \tau_k(f; \psi(t))_{r,p} \leqslant c(k, w)\, \tau_k(f; \psi(t))_{q,p}.$$

Proof. The statement is an immediate consequence of (2.4) and Theorem 5.3. ▌

In other words all moduli τ_k are equivalent to one another when the index of the local modulus q runs between 1 and p. It should be mentioned that the moduli corresponding to different q's, $q \geqslant p$, are not equivalent.

Another consequence of Theorem 5.3 is the quasi-monotonicity of the moduli with respect of w and t, i.e.,

THEOREM 6.2. *Let ψ and $\bar{\psi}$ be connected by* (3.11) *with w and $\bar{w}$, respectively, where the weights satisfy* (3.9). *If $w \leqslant c\bar{w}$ then*

$$\tau_k(f; \psi(t))_{p,p} \leqslant c(k, w)\, \tau_k(f; \bar{\psi}(t))_{p,p}.$$

THEOREM 6.3. *Let ψ satisfy* (3.11) *for some w satisfying* (3.9), $0 < t \leqslant c(w)$ *and $f \in L_p + W_p^k(w)$. Then there is a nondecreasing function of t (for example the weighted K-functional) which is equivalent to $\tau_k(f; \psi(t))_{p,p}$.*

The following theorem is an analog of the well-known property of the moduli of smoothness $\omega_k(f; \lambda t)_p \leqslant c(k)\, \lambda^k \omega_k(f; t)_p$.

THEOREM 6.4. *Let ψ satisfy* (3.11) *for some w satisfying* (3.9), $\lambda > 1$ *and* $0 < \lambda t \leqslant c(w)$. *Then*

$$\tau_k(f; \psi(\lambda t))_{p,p} \leqslant c(k, w)\, \lambda^k \tau_k(f; \psi(t))_{p,p}.$$

The proof follows from Theorem 5.3 and the trivial K-functional property $K(\lambda t, f) \leqslant \lambda K(t, f)$ for $\lambda > 1$.

At the end we give formuli for calculating the magnitude of the moduli of locally differentiable functions. By analogy with Theorem 4.2 for $p < \infty$, w satisfying (3.9), w_1 and w_2 of Type 1 or 2, and ψ given by (3.11) we have

$$\begin{aligned}\tau_k(f; \psi(t))_{p,p[0,1]} \leqslant c(k, w) \Bigg\{ &\left[\int_0^{u_1(t)} |y^k f^{(k)}(y)|^p \, dy \right]^{1/p} \\ &+ t^k \left[\int_{u_1(t)}^{1-u_2(t)} |w(y)^k f^{(k)}(y)|^p \, dy \right]^{1/p} \\ &+ \left[\int_{1-u_2(t)}^{1} |(1-y)^k f^{(k)}(y)|^p \, dy \right]^{1/p} \Bigg\}. \end{aligned} \tag{6.1}$$

For $p = \infty$, w satisfying (3.9), and ψ given by (3.11) we have

$$\begin{aligned}&\tau_k(f; \psi(t))_{\infty,\infty[0,1]} \\ &\quad \leqslant c(k, w) \{ \omega_k(f; u_1(t))_{\infty[0, cu_1(t)]} \\ &\qquad + t^k \sup\{ |w(y)^k f^{(k)}(y)| : y \in [u_1(t), 1-u_2(t)] \} \\ &\qquad + \omega_k(f; u_2(t))_{\infty[1-u_2(t), 1]} \} \end{aligned} \tag{6.2}$$

and we can use Theorem 4.2 for evaluating the moduli of smoothness in this formula.

Using (6.1) and (6.2) for $w(x) = x^\alpha$, $-\infty < \alpha < 1$, $f(x) = x^\beta$, $\beta > -1/p$, $0 < x < 1$, we have

$$\begin{aligned}&\tau_k(f; \psi(t))_{p,p[0,1]} \\ &\quad \leqslant c(k, \alpha, \beta) \begin{cases} t^{(bp+1)/(p-\alpha p)}, & \text{if } \beta + 1/p < (1-\alpha)k; \\ t^k |\log t|^{1/p}, & \text{if } \beta + 1/p = (1-\alpha)k; \\ t^k, & \text{if } \beta + 1/p > (1-\alpha)k. \end{cases}\end{aligned}$$

In all cases above except $\beta = 0, 1, \ldots, k-1$, the τ modulus is actually equivalent to the quantity from the right-hand side because all finite differences of f of order k have one and the same sign.

7. Functional Spaces Generated by the Weighted K-Functional

Let us denote with $B_p^{s,q}(w)$ the interpolation space obtained by the real s/k, q interpolation between the spaces L_p and $W_p^k(w)$, i.e.,

$$B_p^{s,q}(w)=\left\{f\in L_p+W_p^k(w):\left[\int_0^1 t^{-sq-1}K(t^k,f;L_p,W_p^k(w))^q\,dt\right]^{1/q}<\infty\right\}.$$

$B_p^{s,q}(1)$ are the usual homogeneous Besov spaces and so $B_p^{s,q}(w)$ may be called weighted Besov spaces. The aim of this section is to establish, when possible, proximate, embedding results of the type $B_p^{s_1,q}(w_1)\subset B_p^{s_2,q}(w_2)$.

For simplicity we deal only with weights of Types 1, 2, and 3 in $[a,b]=[0,1]$ in this section. Let w_1 and w_2 be not equivalent and let $w=w_1/w_2$ be non-decreasing. Then obviously

$$K(t^k,f;L_p,W_p^k(w_1))\leqslant \max\{1;w(1)\}\,K(t^k,f;W_p^k(w_2)) \tag{7.1}$$

and hence $B_p^{s,q}(w_2)\subset B_p^{s,q}(w_1)$.

Now we shall invert the direction of the inequality in (7.1) changing the argument of one of the K-functionals. We have $w(0)=0$ because w_1 and w_2 are not equivalent. Let w_1 and w_2 be of Type 1 or 2 (for w_1 of Type 3 see Example 2), let w_1 satisfy (3.2) being of Type 1, and let v_j, u_j be the functions associated with w_j ($j=1,2$). We set

$$N(t)=v_1(u_2(t)). \tag{7.2}$$

Theorem 7.1. *Under the above assumptions we have*

$$\begin{aligned}K_2=K(t^k,f;L_p,W_p^k(w_2))&\leqslant c(k,w_1,w_2)\,K(N(t)^k,f;L_p,W_p^k(w_1))\\&=c(k,w_1,w_2)\,K_1.\end{aligned}$$

Proof. It follows from (7.2) that $u_1(N(t))=u_2(t)$. Let g be the function from Lemmas 5.1 or 5.2 corresponding to f, w_1, and $N(t)$ instead of w and t. Then we have

$$g^{(k)}(x)=0 \qquad \text{for} \quad x\in[0,\varepsilon u_2(t)] \qquad (\varepsilon=c(w_1)); \tag{7.3}$$

$$\|f-g\|_{p[0,1]}\leqslant c(k,w_1)\,\tau_k(f;\psi_1(N(t)))_{1,p[0,1]}; \tag{7.4}$$

$$\|\psi_1(N(t))^k g^{(k)}\|_{p[0,1]}\leqslant c(k,w_1)\,\tau_k(f;\psi_1(N(t)))_{1,p[0,1]}, \tag{7.5}$$

where $2k\psi_1(N(t),x)=N(t)\,w_1(x+u_1(N(t)))=N(t)\,w_1(x+u_2(t))$.

Let $\lambda\geqslant 1$. From Property 1.1 and (3.4) we have $w_1(\lambda x)\leqslant \lambda w_1(x)$ or $w_1(\lambda x)\leqslant w_1(x)$ when w_1 is of Type 1 or 2 and $w_2(\lambda x)\geqslant w_2(x)$ and $w_2(\lambda x)\geqslant c(\lambda,w_2)\,w_2(x)$ when w_2 is of Type 1 or 2. Therefore $w(\lambda x)\leqslant c(\lambda,w_2)\,w(x)$

and for every $x > \varepsilon u_2(t)$ we have ($\lambda = \max\{1, 1/\varepsilon\}$) $w(x) \geqslant w(\varepsilon u_2(t)) \geqslant w(u_2(t)/\lambda) \geqslant c(w_1, w_2)\, w(u_2(t))$ and $w(u_2(t)) = w_1(u_2(t))/w_2(u_2(t)) = v_2(u_2(t))/v_1(u_2(t)) = t/N(t)$. Therefore

$$tw_2(x) \leqslant c(w_1, w_2)\, N(t)\, w_1(x) \qquad \text{for} \quad x > \varepsilon u_2(t).$$

Combining (7.3), (7.5), and the above inequality we get

$$\begin{aligned} t^k \|w_2^k g^{(k)}\|_{p[0,1]} &\leqslant c(w_1, w_2)\, N(t)^k \|w_1^k g^{(k)}\|_{p[0,1]} \\ &\leqslant c(k, w_1, w_2) \|\psi_1(N(t))^k g^{(k)}\|_{p[0,1]}. \end{aligned} \tag{7.6}$$

Now from (1.1), (7.4), (7.5), (7.6), and Theorem 4.3 applied for w_1 and $N(t)$ instead of w and t we obtain

$$\begin{aligned} K_2 \leqslant \|f - g\|_p + t^k \|w_2^k g^{(k)}\|_p &\leqslant c(k, w_1, w_2)\, \tau_k(f; \psi_1(N(t)))_{1,p} \\ &\leqslant c(k, w_1, w_2)\, K_1. \quad \blacksquare \end{aligned}$$

COROLLARY 7.1. *Let $-\infty < \alpha < \beta < 1$ and let $\sigma = (1-\beta)/(1-\alpha)$. Then*

$$K(t^k, f; L_p, W_p^k(x^\alpha)) \leqslant c(k, \alpha, \beta)\, K(t^{k\sigma}, f; L_p, W_p(x^\beta)).$$

Proof. We have $w_1(t) = t^\beta$, $v_1(t) = t^{1-\beta}$, $w_2(t) = t^\alpha$, $v_2(t) = t^{1-\alpha}$, $u_2(t) = t^{1/(1-\alpha)}$, and $N(t) = t^\sigma$. Now we obtain the corollary from Theorem 7.1. $\blacksquare$

Combining (7.1) with Corollary 7.1 we obtain

COROLLARY 7.2. Let $-\infty < \alpha < \beta < 1$ *and let $\sigma = s(1-\beta)/(1-\alpha)$. Then*

$$B_{p,q}^s(x^\alpha) \subset B_{p,q}^s(x^\beta) \subset B_{p,q}^\sigma(x^\alpha).$$

EXAMPLE 1. For fixed α, β, and p the embeddings in Corollary 7.2 cannot be improved in terms of Besov spaces. We shall give the examples only for the case $p = q = \infty$. Let $0 < \gamma < 1$ and let $f(x) = (1-x)^\gamma$. Then from (6.2) we get that the τ moduli for both weights are equivalent to t^γ. This shows that the first embedding is sharp. Now let $0 < \gamma < 1$, $k(1-\beta) > \gamma$, and $f(x) = x^\gamma$. Then from (6.2) we get that τ moduli for the weights x^α and x^β are equivalent to $t^{\gamma/(1-\alpha)}$ and $t^{\gamma/(1-\beta)}$, respectively; i.e., the second embedding cannot be improved.

EXAMPLE 2. From $K(t^k, f; L_p, W_p^k(w_1)) = O(t^k)$ and w_1 of Type 3 it does not follow that $K(t^k, f; L_p, W_p^k(w_2)) < \infty$. Let $f_k(x) = x^{-k}$, $w_1(x) = x^2$. Then $K(t^k, f; L_\infty, W_\infty^k(w_1)) \leqslant t^k \|w_1^k f_k^{(k)}\|_\infty = c(k)\, t^k$ but $\tau_k(f_k; tw_2)_{\infty,\infty} = \infty$ provided $w_2(x)\, x^{-2} \to \infty$ when $x \to 0$.

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